1 Problem Set #3

1. Show that the following two expressions for the von Neumann entropy are equivalent:

\[
\lim_{n \to 1} \frac{1}{1 - n} \ln \text{tr}(\rho^n) = (1 - \beta \partial_{\beta}) \ln Z|_{\beta=2\pi}
\]  
(1)

2. The “heat kernel regulated” effective action on a smooth spacetime is:

\[
S_{\text{eff}} = -\frac{1}{2} \int_{\epsilon^2}^{+\infty} \frac{e^{-sm^2}}{(4\pi s)^{d/2}} \left[ \frac{c_0}{s} + c_1 R + \mathcal{O}(s) \right] ds
\]  
(2)

where \(m\) is the mass, \(\epsilon\) is a regulator with units of distance, and \(c_1 = 1/6\) for a minimally coupled scalar field. (The effective action is not local, but the nonlocal aspects are nonperturbative in \(s\) and therefore do not appear as terms in this expansion, nor do they contribute to UV divergences.) Use this regulator to calculate the divergent part of the entanglement entropy in a Rindler wedge in 4-dimensional Minkowski spacetime.

3. Consider the left-moving sector \(\phi_L(v)\) of a free massless scalar field in 1+1 dimensional Minkowski spacetime, \(v\) being a null coordinate. Consider a coherent state formed by acting on the vacuum state \(|0\rangle\) with the exponential of the chiral field operator, times a test function \(f\):

\[
|\Psi\rangle = \exp \left[ \epsilon \int_{t=0}^{x} f(x) \nabla_v \phi_L dx \right] |0\rangle,
\]  
(3)

Use the replica trick to calculate the \(\mathcal{O}(\epsilon^2)\) piece of the entropy in the interval \(x = [0, +\infty]\), at time \(t = 0\). (What happens if \(f\) is chosen to be imaginary?) If you do everything correctly, \(\delta S\) should vanish. This is because the state produced by the exponential turns out to be a coherent state, whose entropy vanishes due to the unitary invariance property of the von Neumann entropy.