1 Problem Set #1

1. Suppose that $\rho = e^{-\beta H}$ is a thermal state in an ordinary Hilbert space, where $H$ is any operator for which the resulting state is normalizable. Show that this state satisfies the KMS condition for any operators $A, B$:

$$\langle A(\tau)B \rangle_\rho = \langle BA(\tau + i\beta) \rangle_\rho,$$

where $A(\tau)$ represents the time translation of the operator $A$, and $\langle BA(\tau) \rangle_\rho$ is analytic in the strip $0 \geq \text{Im}(z) \geq \beta$.

(This exercise justifies the use of the KMS condition as the definition of a thermal state in more exotic contexts, such as type III von Neumann algebras.)

2. Consider the following properties of the von Neumann entropy described during the lecture:

i) Positivity: $S(\rho) \geq 0$,

ii) Invariance under Unitaries: $S(U\rho U^\dagger) = S(\rho)$,

iii) Additivity under Tensor Product: $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$,

iv) Triangle Identity: $S(A) + S(B) \geq S(AB) \geq |S(A) - S(B)|$,

v) Continuous (for finite dimensional Hilbert spaces),

vi) {Strong Subadditivity: $S(AB) + S(BC) \geq S(ABC) + S(B)$},

vii) Concavity: $S(\lambda \rho + (1 - \lambda)\sigma) \geq \lambda S(\rho) + (1 - \lambda)S(\sigma)$,

viii) Chain Rule: If $\rho = \otimes_i \lambda_i \rho_i$ (block diagonal), then $S(\rho) = \langle S(\rho_i) \rangle - \sum_i \lambda_i \ln \lambda_i$.

(a) Show that Strong Subadditivity implies Weak Monotonicity: $S(AB) + S(BC) \geq S(A) + S(C)$ by assuming the existence of a 4th system $D$ such that $ABCD$ is pure, and using the fact that $S(R) = S(\overline{R})$.

(b) See how many of the properties you can prove on your own. (But don’t try too hard for Strong Subadditivity, whose proof is very difficult!) For the proof of Araki-Lieb, try using the purifier trick described in (a). Note that Concavity can be proven from Subadditivity and the Chain Rule.

(c) Determine which of these properties are also obeyed by the Renyi entropy $S_n = \frac{1}{1-n} \ln \text{tr}(\rho^n)$, which limits to the von Neumann entropy as $n \to 1$. For simplicity you may wish to focus on the case $n > 1$ and finite dimensional Hilbert spaces. Note: the simplest counterexample to Concavity involves a Hilbert space with a large number of states.