1 Problem Set #1

1. Suppose that $\rho = e^{-\beta H}$ is a thermal state in an ordinary Hilbert space, where H is any operator for which the resulting state is normalizable. Show that this state satisfies the KMS condition for any operators A, B:

$$\langle A(\tau)B\rangle_\rho = \langle BA(\tau + i\beta)\rangle_\rho, \tag{1}$$

   where $A(\tau)$ represents the time translation of the operator A, and $\langle BA(z)\rangle_\rho$ is analytic in the strip $0 \geq \mathrm{Im}(z) \geq \beta$.

   (This exercise justifies the use of the KMS condition as the *definition* of a thermal state in more exotic contexts, such as type III von Neumann algebras.)

2. Consider the following properties of the von Neumann entropy described during the lecture:

   i) Positivity: $S(\rho) \geq 0$,

   ii) Invariance under Unitaries: $S(U\rho U^\dagger) = S(\rho)$,

   iii) Additivity under Tensor Product: $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$,

   iv) Triangle Identity: $S(A) + S(B) \geq S(AB) \geq |S(A) - S(B)|$,

   v) Continuous (for finite dimensional Hilbert spaces),

   vi) {Strong Subadditivity: $S(AB) + S(BC) \geq S(ABC) + S(B)$},

   vii) Concavity: $S(\lambda\rho + (1-\lambda)\sigma) \geq \lambda S(\rho) + (1-\lambda)S(\sigma)$,

   viii) Chain Rule: If $\rho = \otimes_i \lambda_i \rho_i$ (block diagonal), then $S(\rho) = \langle S(\rho_i)\rangle - \sum_i \lambda_i \ln \lambda_i$.

   (a) Show that Strong Subadditivity implies Weak Monotonicity: $S(AB) + S(BC) \geq S(A) + S(C)$ by assuming the existence of a 4th system D such that $ABCD$ is pure, and using the fact that $S(R) = S(\bar{R})$.

   (b) See how many of the properties you can prove on your own. (But don't try too hard for Strong Subadditivity, whose proof is very difficult!) For the proof of Araki-Lieb, try using the purifier trick described in (a). Note that Concavity can be proven from Subadditivity and the Chain Rule.

   (c) Determine which of these properties are *also* obeyed by the Renyi entropy $S_n = \frac{1}{1-n} \ln \mathrm{tr}(\rho^n)$, which limits to the von Neumann entropy as $n \to 1$. For simplicity you may wish to focus on the case $n > 1$ and finite dimensional Hilbert spaces. *Note: the simplest counterexample to Concavity involves a Hilbert space with a large number of states.*