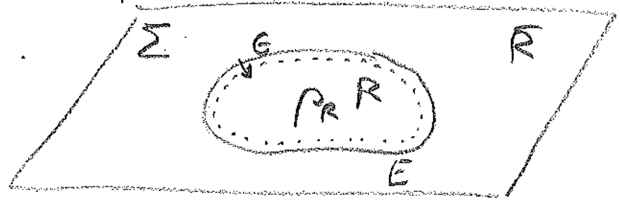
ENTANGLEMENT ENTROPY

Defined as $S(\rho_R) = -\mathrm{tr}(\rho_R \ln \rho_R)$



depends on choice of

(i) QFT

(ii) region R

(iii) state Ψ of system – via restriction to ρ_R

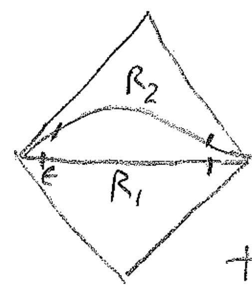
– UV divergent, depends on short distance cutoff ϵ.

(Sometimes also IR divergent)

– Should have properties similar to QM

(modulo cutoff issues)

- $S(R) = S(\bar{R})$ (w/ "same" cutoff on both sides)
- $S(R_1) = S(R_2)$ if $D[R_1] = D[R_2]$



Cutoff can mess this up in thy w/ grav anomaly!

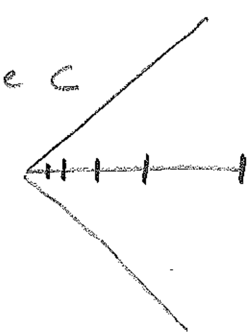
EXPECTED DIVERGENCE STRUCTURE

D=2 CFT

$$S = \frac{c}{3}\ln\left(\frac{r}{\epsilon}\right) + \text{finite}$$ for length r, central charge c

$\frac{c}{3}$: universal

finite: scheme dependent, state dependent

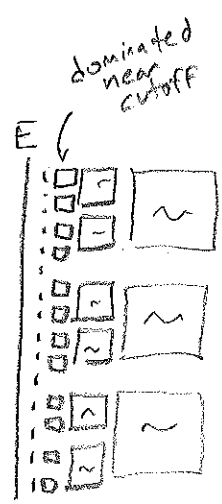


D > 2: AREA LAW

$$S = \# \frac{A}{\epsilon^{D-2}} + \text{subleading} + \text{finite}$$

subleading: structure can be deduced by dim. analysis

R, K^2, m^2... look for products w/ weight up to $D-2$



Zoom in on E — close to Rindler

$\int_\epsilon f(\lambda)\, d\lambda$ = power + log

INTEGRATION OF EE

Assume that:

1) Contributions to S from scales $\lambda_1 \gg \lambda_2$ factorize & are independent.

2) In UV, contribution is local integral on boundary which depends in a scale-invariant smooth way on:
   (a) ϵ
   (b) local geometrical quantities (e.g. R, K^2, ..)
   (c) a set of relevant couplings (e.g. m^2) w/ def. scaling Δ

3) Contributions from $\lambda \gg \epsilon$ are UV finite & don't care about choice of cutoff (ϵ_1 vs. ϵ_2)

#1 & #2 allow us to integrate power laws from $\lambda \sim R$ to $\lambda \sim \epsilon$:

[NOTE: EXCEPTIONS TO ASSUMPTIONS EXIST]

NONLOCAL

$$S = \sum_p a_p \int_\epsilon p^{[w \le D-2]} dA \, \epsilon^{D-2-w} + \sum a_p \int_\epsilon p^{[w=D-2]} dA \ln(\epsilon) + S_{finite}(p)$$

SCHEME-DEPENDENT

UNIVERSAL (usually only exists in D = even)

UNIVERSAL UP TO LOCAL COUNTERTERMS (usually none in D = odd)

#3 tells us any ϵ power law (incl. ϵ^0)

4D EE

free field of mass m [and reasonable ϵ]

$$S = \# \frac{A}{\epsilon^2} + \left(\alpha \int R \, dA + \frac{\beta}{2} \int \left(R_i^{\ i} - \frac{1}{2} K_i^{\ i(a)} K^{i}_{\ j(a)} \right) dA + \gamma \left(R_{ij}^{\ \ ij} - K_{ij}^{\ \ (a)} K^{ij}_{\ \ (a)} \right) + \zeta m^2 A \right) \ln \epsilon \ + \text{ finite}$$

(i, j transverse)

where R is curvature tensor (vanishes in Mink.), $K_{ij}^{(a)}$ extrinsic curvature

$\alpha, \beta, \gamma, \zeta$ are universal quantities depending on the thy.

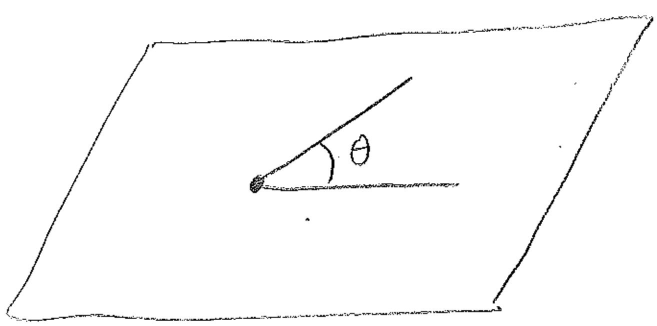
For a CFT, depends on central charges a, c:

$$\boxed{\begin{aligned} \alpha &= (a + \tfrac{1}{3}c)/2 \\ \beta &= -2a - c \\ \gamma &= (a+c)/2 \\ \zeta &= 0 \end{aligned}} \longrightarrow \left[c \left(C_{ijij} + K^2 \text{ terms} \right) - a^{D-2}R \right] \ln \epsilon$$

3D EE

$$S = \# \frac{A}{\epsilon} + \text{ finite}$$

Normally no log div, but can appear w/ sharp corners since θ is dimensionless!



all this assumes no anomalous dimensions

5D

$$S = \# \frac{A}{\epsilon^3} + \cdots \frac{1}{\epsilon} + \text{ finite}$$

6D

$$S = \# \frac{A}{\epsilon^4} + \cdots \frac{1}{\epsilon^2} + \cdots \ln \epsilon + \text{ finite}$$

Strategies to deal w/ scheme dependence:

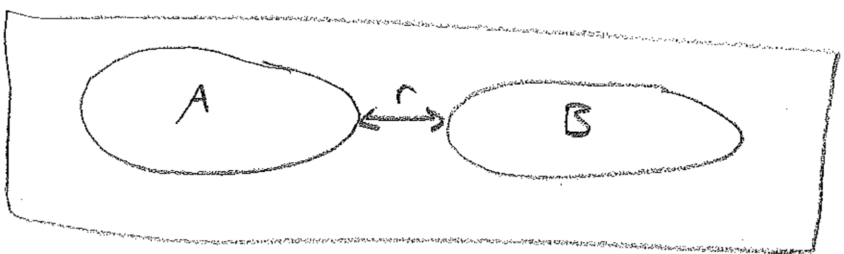
A) just pick a regulator and stick w/ it.

B) focus on universal quantities (more to come)

C) renormalize into local counterterms (in quantum gravity)

D) only consider entropy differences (if no state dependence)

MUTUAL INFORMATION



useful because sensitive to ALL degrees of freedom, not just local O's.

$I_{A,B}$

$$\equiv S_A + S_B - S_{AB} = S(\rho_{AB} | \rho_A \otimes \rho_B)$$

local divergences cancel if A, B separated!

measures degree to which A, B are correlated

UNIVERSAL

positive, monotonic under growing A or B

equivalent.

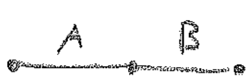
STRONG SUBADDITIVITY

diverganges often cancel (not always)

$$S_{AB} + S_{BC} \geq S_{ABC} + S_B$$

$$\cup \qquad \cap$$

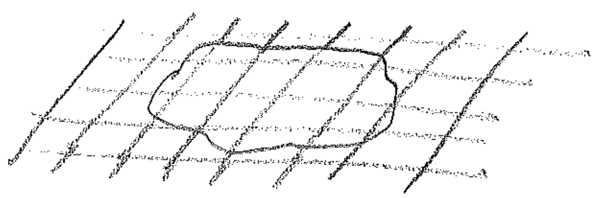
plain subadditivity trivial for touching regions



$$S_A + S_B - S_{AB} = \infty$$

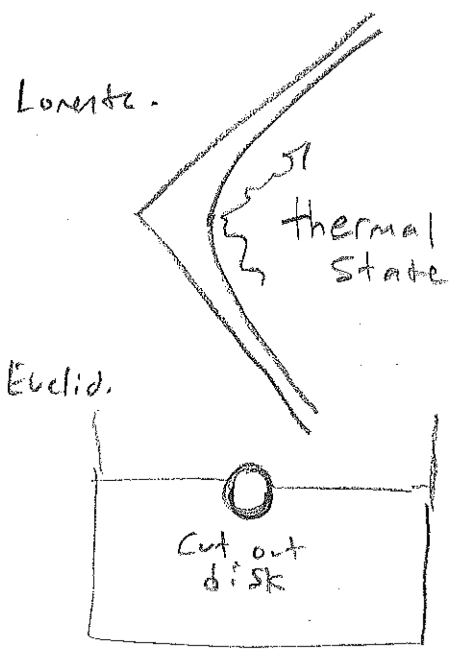
REGULATOR EXAMPLES? (1.10)

(a) Lattice



literal VN entropy of region

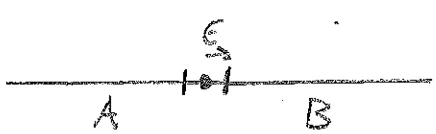
(b) t'Hooft Brick Wall



put b.c. on brick wall

must show physics is unaffected far from the wall

(c) Mutual Info w/ small gap



$$S_A^{(\epsilon)} \equiv \frac{I_{A,B}}{2} = \frac{S_A + S_B - S_{AB}}{2}$$

S_A? 0 in pure state

purely passive

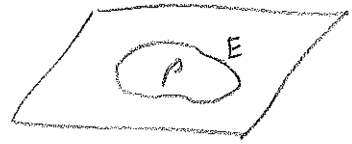
not clearly equivalent

(d) Replica trick — to come!

REVIEW

Last time we defined EE and related concepts

$$S = -\mathrm{tr}(\rho \ln \rho)$$



UV divergent

Advantages

- Sensitive to <u>all</u> dof
- tells us about nonlocal obs.
- related to entanglement, thermo etc.
- many nice inequalities / relations

Disadvantages

- sensitive to all dof, nonlocal
- almost impossible to directly measure in exp.
- difficult (but possible) to calculate theoretically

today I will show you how!

C-THEOREM

- Lorentz Inv.
- Unitary
- 2D

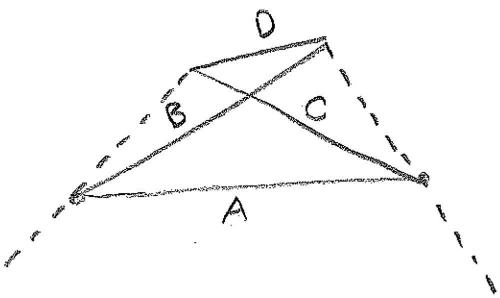
$\Longrightarrow$ "c" is a nondecreasing fn of RG flow

where
- c = central charge for CFTs
- c stationary @ fixed points

(c not unique!)

proved by Zamoldchikov using 2pt fn of $T_{\mu\nu}$

easier proof w/ EE (Casini-Huerta)



lengths obey

$A \cdot D = B \cdot C$ by relativistic geometry

(So if $A = \lambda B$, $C = \lambda D$

SSA:

$$S(B) + S(C) \geq \underset{\displaystyle S(A)}{S(B \cup C)} + \underset{\displaystyle S(D)}{S(B \cap C)}$$

rewrite as

$$S(B) - S(D) \geq S(\lambda B) - S(\lambda D) \qquad \lambda \geq 1$$

For 1+1 CFT in vacuum, Lorentz sym. says S only depends on proper length r:

$$S(r) = \frac{c}{3} \ln\left(\frac{r}{\epsilon}\right) + \text{const.}$$

So $\underbrace{r \frac{d}{dr} S(r)}_{\text{decreasing under RG flow}} = \frac{c}{3}$ for CFT

$$\left[\frac{c'(r)}{3} = r S''(r) + S'(r) \leq 0 \right]$$

not same as Z's c-function

Can this be extended to prove analogue of c-thm in higher D?

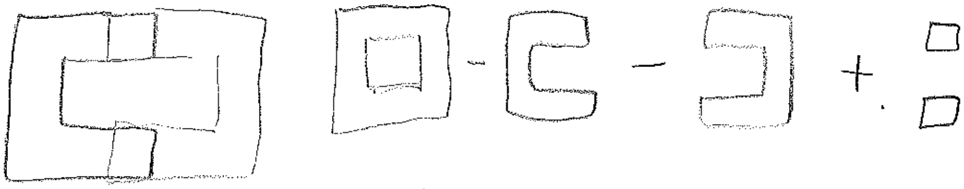
3d — F theorem — proof from EE

4d — a theorem — proof from 4 pt fn of $T_{\mu\nu}$, progress to EE derivation

TOPOLOGICAL EE

Consider a topologically ordered phase of matter. TQFT in IR
e.g. Chern-Simons in (2+1) d

no local dof by definition, yet has nonzero EE



≡ 2 0 0

2 disks − 1 − 1 + 2

[DISCUSS EDGE MODES — DIGRESS TO YM?]

$$S_{disk} = \underbrace{\frac{\ell}{\epsilon}} - \gamma + \mathcal{O}\left(\frac{1}{\ell}\right)$$

- nonuniversal,
- depends on UV microphysics,
- not topological

γ: universal, nonlocal

$\gamma = \ln(D)$ where D = "quantum dimension" e.g. # elements of discrete YM

$\gamma > 0$ in vacuum for good field theories

Can diagnose topological phases!

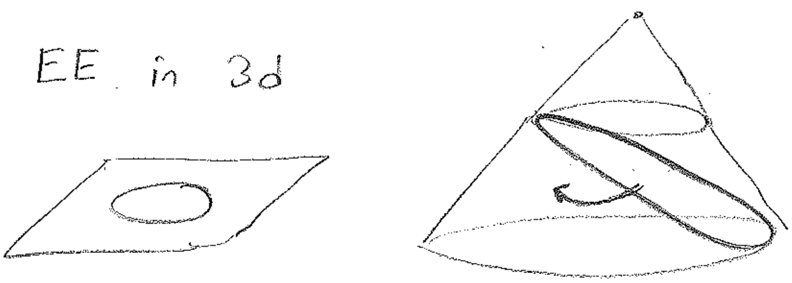
related to free energy on S_3 in CFT

$$F = -\ln Z_{(S_3)}$$

SWITCH THESE?

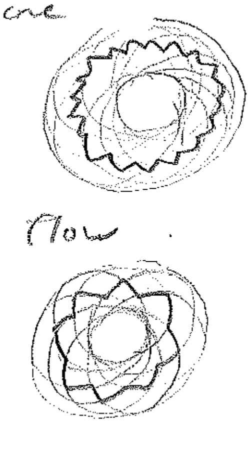
F-THEOREM

EE in 3d



tilted circles in lightcone
$$t^2 = x^2 + y^2$$

CASINI-HUERTA



SSA shows that $\underbrace{r S'(r) - S(r)}_{= \gamma = F \text{ @ fixed point}}$ decreases under RG flow
$[S''(r) \leq 0]$

F	γ
topological	$\ln D$
light Dirac	.219
light (noncompact) ϕ	.0638

free fields: light Dirac, light (noncompact) ϕ

Tarun Grover