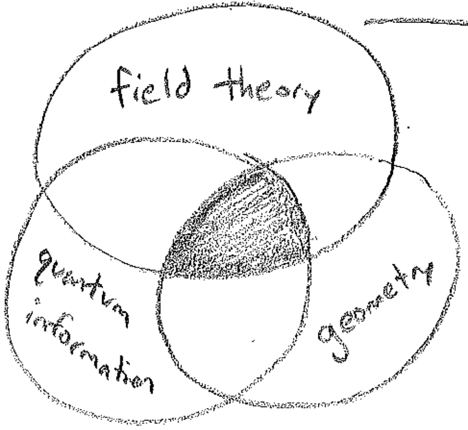
BIG PICTURE



QFT: learn about new observables which allow one to probe non-pointlike aspects of the field theory

QG: Can we reconstruct spacetime from information theory?

Deep relationships between Einstein gravity & thermodynamics. Planckian microphysics, holography.

Qinfo: generalization to continuum

new ways to represent entanglement structures

hints at new inequalities?

Mixed States

QM of a subsystem

Let $H = H_1 \otimes H_2$ be a Hilbert Space factorization.

Let Ψ = pure but entangled state of H.

Restrict to only H_1, use mixed state:

$$\rho_1 = tr_2(|\Psi\rangle\langle\Psi|) = \begin{pmatrix} \cdot & & \\ & \ddots & \\ & & \cdot \end{pmatrix}$$

Can diagonalize via U

normalized $tr(\rho) = 1$.

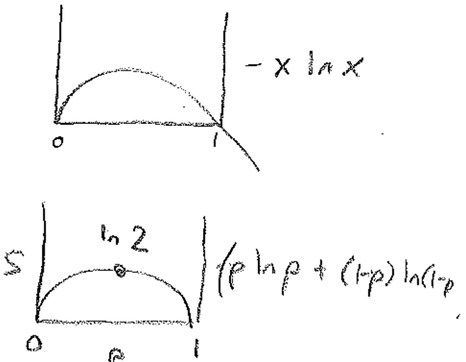
Von Neumann Entropy

Particularly nice attribute is vN entropy

$$S(\rho) = -tr(\rho \ln \rho)$$

$= 0$ for pure

$= \ln D$ for max mixed $\rho = \frac{1}{D}$ in D-dim H



Properties:

1) Positive: $S(\rho) \geq 0$

2) Inv. under unitaries: (a) $S(U\rho U^\dagger) = S(\rho)$, (b) adding extra $p=0$ states

3) Additive: $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$ def $S(A) \equiv S(\rho_A)$

4) Triangle: $S(A) + S(B) \underset{\uparrow}{\geq} S(AB) \underset{\uparrow}{\geq} |S(A) - S(B)|$

(a) Subadditivity (b) Araki-Lieb

5) Continuous for finite D. (Lower semicontinuous for ∞-D: if $\rho = \lim \rho_n$, $S(\rho) \leq \lim\inf S(\rho_n)$)

[2, 3, 4a, 5 uniquely pick out S up to scaling]

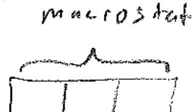
6) Strong Subadditivity: $S(AB) + S(BC) \geq S(ABC) + S(B)$

$\geq S(A) + S(C)$ (a.k.a. week mono)

7) Concavity: Let $\sum_i \lambda_i = 1$, then $S(\sum_i \lambda_i \rho_i) \geq \sum_i \lambda_i S(\rho_i)$

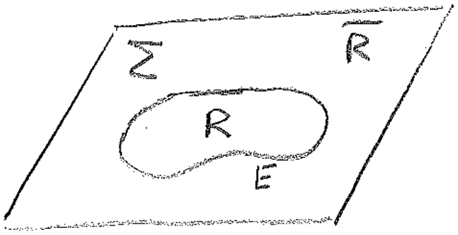
8) Chain Rule: Let $\rho = \underset{i,\text{ diagonal}}{\overset{\text{block}}{\bigoplus}} \lambda_i \rho_i$, then $S(\rho) = \langle S(\rho_i) \rangle_\lambda - \sum_i \lambda_i \ln \lambda_i$

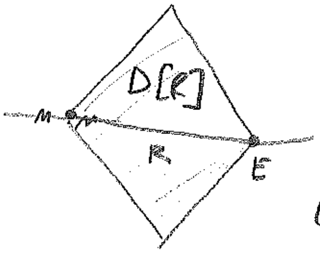
Strong Additivity



States of QFT Regions

pick a timeslice Σ of flat or curved spacetime
divide into 2 pieces w/ codim 2 surface E





Define $D[R]$ as region causally determined by R
(pts s.t. all causal curves pass through R)

Suppose we restrict only to $\mathcal{O}$'s measurable in $D[R]$
(not same as $\mathcal{O}$'s on R since many $\mathcal{O}$'s require time smearing in QFT!)
Generates algebra $A[R]$.

Strictly speaking, no $\mathcal{H}$ defined in a region w/ boundary, due to ∞ many d.o.f. entangled near E!
(No pure states, so no density matrix.)

HIGH ROAD: reformulate all physics in terms of algebras. (AQFT)

LOW ROAD: put a UV cutoff @ short distances, to make $\mathcal{H}$ exist.

ALGEBRAS

Can define subalgebra $A \in A(\mathcal{H}_{QFT})$ as collection of $\mathcal{O}$'s that

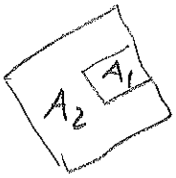
(i) includes trivial $\mathbb{C}$

(ii) closed under $+$, $\times$, $*$ (adjoint)

(iii) nice closure properties – C^* algebra closed wrt norm topology.

VN algebra $A = A''$ (weak topology)

- a state ρ is defined as a linear functional on A
  i.e. via expectation values $\langle \mathcal{O} \rangle_\rho$, $\mathcal{O} \in A$
- if $A_1 \subset A_2$, $\rho(A_1) \equiv \rho(A_2)$ restriction is trivial!



- for finite systems, basically equivalent to $\mathcal{H}$ picture, except allows "superselection" $\mathcal{O}$'s that commute w/ everything

algebra $\Longleftrightarrow$ set of $\mathcal{H}$'s. $A = A(H_1) \oplus A(H_2) \oplus A(H_3)$...

- for ∞ systems, can be quite different.

vN algebra classification [HYPERFINITE – can be approx by sequence of finite A's]
FACTORS – no superselection, can take sums & integrals

I_n $A(H)$ for n-dim H

I_∞ " for ∞-dim H 0 temp (+ finite excitations)

II_1 e.g. ∞ qubits @ ∞ temp. (+ finite exc.)

$II_\infty = II_1 \otimes I_\infty$

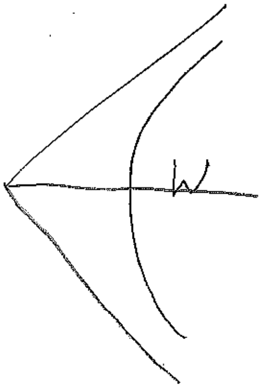
III e.g. ∞ system @ finite temp + finite exc. NO TRACE! $\mathrm{tr}\,\mathcal{O} = ?$

has subclassification but "generic" is III_1 —— QFT of region.

THERMALITY OF WEDGE REGIONS

Some specially symmetric regions restrict to thermal states

Consider vacuum $|0\rangle$ restricted to "Rindler wedge"



W = past ∩ future of unif. acc. worldline

Coordinates

$X \geq |t|$

boost energy $K = \int_{X>t} T_{tt}\, x\, dx\, dy\, dz$ @ $t=0$

translation inv. in y, z plane, boost inv. in t-x plane

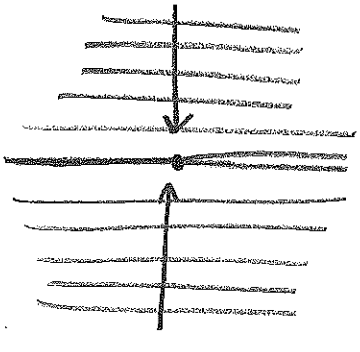
Vacuum is thermal wrt boost energy when restricted to ρ_W — "Unruh effect"
w/ temp $\hbar/2\pi$
[dimensionless quantity]

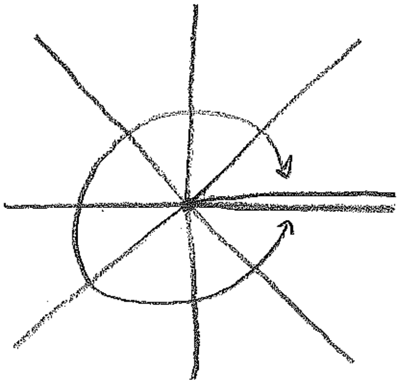
loc. temp. $\frac{\hbar a}{2\pi}$

LOWBROW:

$$\rho_W = \frac{e^{-2\pi K}}{tr(e^{-2\pi K})}$$

Unruh-Weiss derivation





$\rho = tr_{x<0}\, |0\rangle\langle 0|$

$\int D\phi\, e^{-S}$

HIGHBROW:

thermal means KMS state: $\langle A(\tau) B\rangle_\rho = \langle B A(\tau + i\beta)\rangle_\rho$ for $\forall A, B$ [K]

β = inv. temp.
τ = thermal time

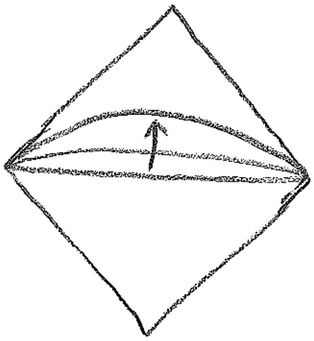
must be able to analytically continue $\langle BA(\tau)\rangle$ in strip of complex plane $0 \leq Im(\tau) \leq \beta$ & must be continuous on boundary

Bisognano-Wichman thm. proves KMS for any QFT if

a) stable wrt normal time translations

b) boost inv. WORKS EVEN IF INTERACTING

OTHER THERMAL REGIONS

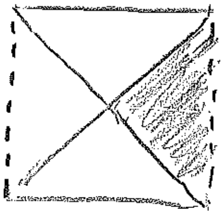


In CFT, diamond is conformal to wedge

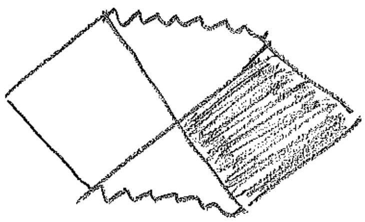
vacuum is thermal wrt symmetry that preserves wedge

$$K = \int_{|x|<R} \frac{R^2 - |\vec{x}|^2}{2R} T_{tt} \, d^{D-1}\vec{x}$$

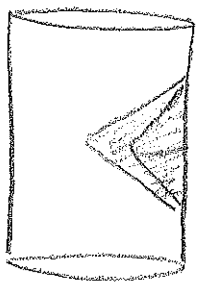
CURVED SPACETIMES



static patch dS in BD = HH vacuum



static BH in HH state



AdS-Rindler in AdS vacuum

MODULAR FLOW

Alternatively, <u>any</u> state σ may be regarded as thermal if we pick $K = \ln \sigma$ (assuming σ has no 0 prob. outcomes)

$e^{iK} \rho e^{-iK}$ is modular flow

Rigorously defined for any "faithful" state algebraically.

RELATIVE ENTROPY

[DEFINED FOR GENERAL ALGEBRA - HYPERFINITE CAN TAKE LIMIT]

$$S(\rho | \sigma) = \mathrm{tr}(\rho \ln \rho) - \mathrm{tr}(\rho \overbrace{\ln \sigma}^{K})$$

[normalization]

free modular energy wrt σ

- positive, 0 iff $\rho = \sigma$, $+\infty$ possible but usually finite even in QFT
- convex $\hookrightarrow$ (1st law $\delta S = \delta \langle K \rangle$ for 1st order change $\rho = \sigma + \delta\rho$ $dE = TdS$)
- monotonic under restrictions of ρ, σ to subalgebra. 

? - lower semicontinuous

ENTANGLEMENT ENTROPY

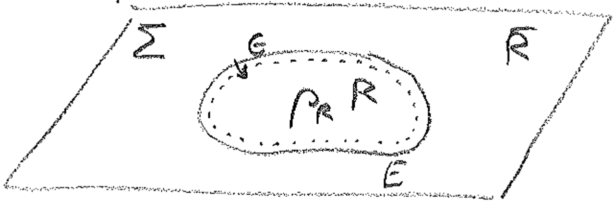
Defined as $S(\rho_R) = -\mathrm{tr}(\rho_R \ln \rho_R)$

depends on choice of

(i) QFT

(ii) region R

(iii) state Ψ of system – via restriction to ρ_R



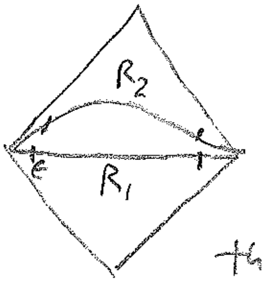
– UV divergent, depends on short distance cutoff ϵ.

(Sometimes also IR divergent)

– Should have properties similar to QM

(modulo cutoff issues)

- $S(R) = S(\bar{R})$ (w/ "same" cutoff on both sides)
- $S(R_1) = S(R_2)$ if $D[R_1] = D[R_2]$



Cutoff can mess this up in thy w/ grav anomaly!

EXPECTED DIVERGENCE STRUCTURE

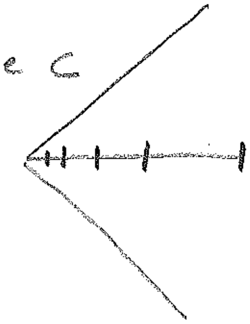
D=2 CFT

$$S = \frac{c}{3} \ln\left(\frac{r}{\epsilon}\right) + \text{finite}$$

for length r, central charge c

↑ Universal

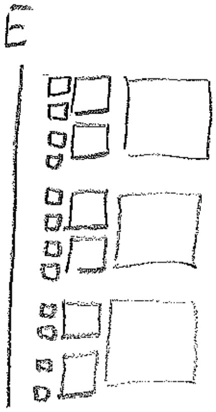
↑ Scheme dependent, State dependent



D > 2: AREA LAW

$$S = \# \frac{A}{\epsilon^{D-2}} + \text{subleading} + \text{finite}$$

↑ structure can be deduced by dim. analysis



Zoom in on E – close to Rindler

$R, K^2, m^2 \ldots$ look for products w/ weight up to $D-2$

$$\int_\epsilon f(\lambda)\, d\lambda = \text{power} + \log$$

INTEGRATION OF EE

Assume that:

1) Contributions to S from scales $\lambda_1 >> \lambda_2$ factorize & are independent.

2) In UV, contribution is local integral on boundary which depends in a scale-invariant smooth way on:
    - (a) ϵ
    - (b) local geometrical quantities (e.g. R, K^2, ..)
    - (c) a set of relevent couplings (e.g. m^2) w/ def. scaling Δ

3) Contributions from $\lambda >> \epsilon$ are UV finite & don't care about choice of cutoff (ϵ_1 vs. ϵ_2)

[NOTE: EXCEPTIONS TO ASSUMPTIONS EXIST]

#1 & #2 allow us to integrate power laws from $\lambda \sim R$ to $\lambda \sim \epsilon$:

$$S = \sum_p \underset{\substack{\uparrow \\ \text{SCHEME-DEPENDENT}}}{a_p} \int_\epsilon \rho^{[w \leq D-2]} dA\, \epsilon^{D-2-w} + \sum \underset{\substack{\uparrow \\ \text{UNIVERSAL} \\ \text{(usually only exists} \\ \text{in D = even)}}}{a_p} \int_\epsilon \rho^{[w = D-2]} dA \ln(\epsilon) + \overset{\substack{\text{NONLOCAL} \\ \downarrow}}{\underset{\substack{\uparrow \\ \text{UNIVERSAL} \\ \text{UP TO} \\ \text{LOCAL} \\ \text{COUNTERTERMS} \\ \text{(usually none} \\ \text{in D = odd)}}}{S_{finite}}}(\rho)$$

#3 tells us any ϵ power law (incl. ϵ^0)

4D EE

free field of mass m [and reasonable ϵ]

$$S = \# \frac{A}{\epsilon^2} + \left(\alpha \int R \, dA + \frac{\beta}{2} \int \left(R_i^{\ i} - \frac{1}{2} K_i^{\ i(a)} K^{i}_{\ j(a)}\right) dA + \gamma\left(R_{ij}^{\ \ ij} - K_{ij}^{(a)} K^{ij}_{(a)}\right) + \zeta m^2 A\right) \ln \epsilon + \text{finite}$$

where R is curvature tensor (vanishes in Mink.), $K_{ij}^{(a)}$ extrinsic curvature

$\alpha, \beta, \gamma, \zeta$ are universal quantities depending on the thy.

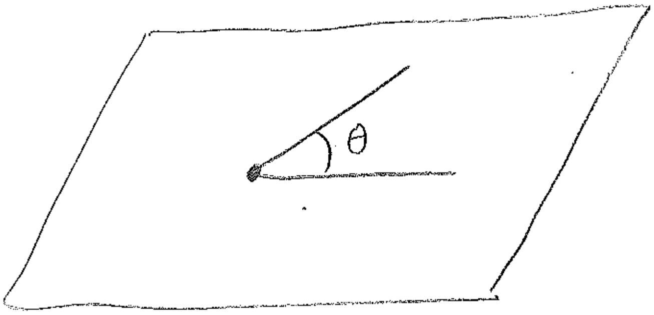
For a CFT, depends on central charges a, c:

$$\boxed{\begin{aligned} \alpha &= (a + \tfrac{1}{3}c)/2 \\ \beta &= -2a - c \\ \gamma &= (a+c)/2 \\ \zeta &= 0 \end{aligned}}$$

3D EE

$$S = \# \frac{A}{\epsilon} + \text{finite}$$

Normally no log div, but can appear w/ sharp corners since θ is dimensionless!



all this assumes no anomalous dimensions

5D

$$S = \# \frac{A}{\epsilon^3} + \cdots \frac{1}{\epsilon} + \text{finite}$$

6D

$$S = \# \frac{A}{\epsilon^4} + \cdots \frac{1}{\epsilon^2} + \cdots \ln \epsilon + \text{finite}$$

Strategies to deal w/ scheme dependence:

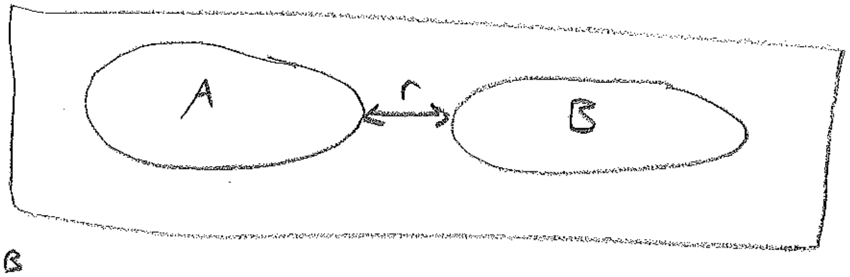
A) just pick a regulator and stick w/ it.

B) focus on universal quantities (more to come)

C) renormalize into local counterterms (in quantum gravity)

D) only consider entropy differences (if no state dependence)

MUTUAL INFORMATION



useful because sensitive to ALL degrees of freedom, not just local O's.

$$I_{A,B} \equiv S_A + S_B - S_{AB} = S(\rho_{AB} | \rho_A \otimes \rho_B)$$

local divergences cancel if A, B separated!

measures degree to which A, B are correlated

UNIVERSAL

positive, monotonic under growing A or B

equivalent.

STRONG SUBADDITIVITY

divergences often cancel (not always)

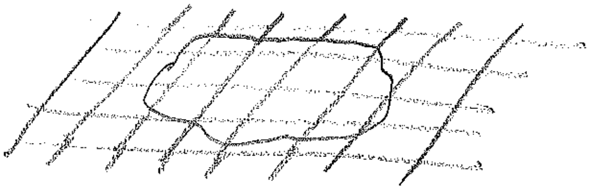
$$S_{AB} + S_{BC} \geq \underset{\cup}{S_{ABC}} + \underset{\cap}{S_B}$$

plain subadditivity trivial for touching regions

$$S_A + S_B - S_{AB} = \infty.$$

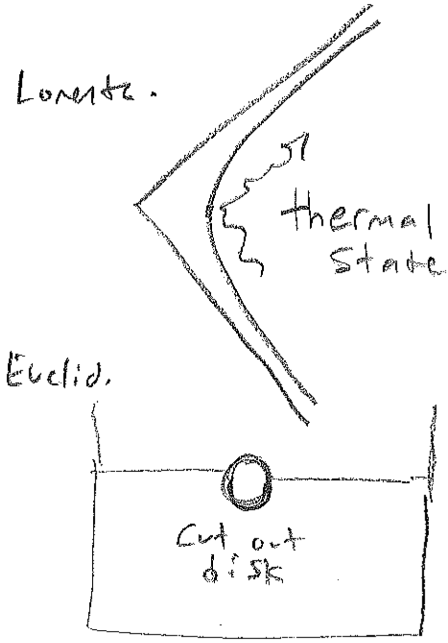
REGULATOR EXAMPLES?

(a) Lattice



literal VN entropy of region

(b) t'Hooft Brick Wall



put b.c. on brick wall

must show physics is unaffected far from the wall

(c) Mutual Info w/ small gap

$$S_A^{(\epsilon)} \equiv \frac{I_{A,B}}{2} = \frac{\overset{\nearrow S_A ?}{S_A + S_B} - \overset{\nearrow 0}{S_{AB}}}{2}$$

purely passive

not clearly equivalent

(d) Replica trick — to come!